JOURNAL OF AIRCRAFT
Vol. 42, No. 6, November–December 2005

Constant Lift Coefficient Turns

G. Iosilevskii*
Technion—Israel Institute of Technology, 32000 Haifa, Israel

Lateral displacement in a U-turn may become a problem when aircraft position has to be accurately maintained, but the excess thrust is inadequate to sustain level turn of sufficiently small radius. It seems plausible that lateral displacement in a U-turn can be diminished at the expense of mechanical energy: This trade of energy is the subject matter of this study. With use of a perturbation technique, closed-form analytical solution has been obtained for the lateral displacement and mechanical energy loss at the exit of the turn, under the assumption that lift coefficient remains constant throughout the turn. The results elucidate intrinsic differences between airspeed-depleting and altitude-depleting turns.

Nomenclature

a	=	combination of n_s and λ_1/λ_s
C_D	=	drag coefficient
$C_{D,p}$	=	parasite (zero lift) drag coefficient
$C'_{D,s}$	=	effective drag coefficient at sustained turn conditions
C_L	=	lift coefficient
$C_{L,\alpha}$	=	lift-slope coefficient
D	=	drag
g	=	acceleration of gravity
h	=	altitude
h_e	=	energy altitude
K	=	induced drag parameter
L	=	lift
M	=	Mach number
n	=	load factor (lift-to-weight ratio)
p	=	static pressure
S	=	wing area
T	=	thrust
t	=	time
v	=	true airspeed
W	=	weight
$\langle x, y, z\rangle$	=	coordinates of the aircraft center of mass
γ	=	specific heats ratio (1.401)
δ	=	perturbation parameter (relative lift or load factor increase)
λ_s	=	combination of n_s and $C'_{D,s}/C_{L,s}$
λ_1	=	combination of n_s and $(\partial C_D/\partial C_L)_s$
ρ	=	density at altitude
ψ	=	azimuth

Subscripts

s	=	at (or corresponding to) sustained level turn at the entrance altitude and airspeed
π	=	at (or corresponding to) the exit of the turn
0	=	at (or corresponding to) the entrance into the turn
1	=	first-order-perturbation with respect to δ (except λ_1)

Introduction

Background

THE problem addressed herein rose during a particular flight test where it was essential to keep station at high-altitude flying with external tanks at low supersonic Mach numbers. Under these circumstances the test aircraft was barely able to keep altitude in a sustained 50-deg bank turn, which made the turn diameter in excess of 10 n mile. The altitude has to be kept fairly accurately, and the obvious question arose how the airspeed on the exit of the turn can be traded to lower the lateral displacement during the turn. Notwithstanding the multitude of known solutions to the large variety of turn optimization problems,[1] no answer was found to that particular question in published results. A solution is suggested later.

The problem of trading altitude, rather then airspeed, for the turn diameter has been addressed by Gates[2] under the assumptions that the altitude change during the turn is small as compared with the turn diameter and the thrust and aerodynamic forces remain constant during the turn. An extension to the Gates solution, allowing for altitude-dependent forces, is suggested later.

Assumptions

The maneuver that will be analyzed throughout the exposition is that of reversing a flight direction (U-turn) in still atmosphere (no wind) either at constant altitude, or at constant airspeed. In the latter case, the angle between the aircraft trajectory and the horizontal plane will be assumed small; additional assumptions pertaining to this case will be specified in due course. In both cases, however, it will be assumed that the lift coefficient is kept constant throughout the turn. This assumption is quite realistic: Fighter pilots commonly rely on an angle of attack (AOA) indicator during energy maneuvering; moreover, a constant AOA roughly corresponds to a constant longitudinal stick position.[3]

It will be assumed that characteristic times on which the load factor and bank angle are established (few seconds)[3] are much shorter than the characteristic time scale (100 s) of the maneuver itself. In turn, the latter will be assumed small as compared with the characteristic time scale on which the aircraft weight changes appreciably (1000 s) (Ref. 4). Consistent with these assumptions, a point mass approximation will be adopted, with the mass remaining constant throughout the turn. To assist the analysis further, the Earth will be assumed flat.

Notation

Select a right-handed coordinate system C with the x axis coinciding with the initial (horizontal) flight path of the aircraft and the z axis pointing toward the Earth; the origin O of C is conveniently located at the aircraft center of gravity at the beginning of the turn; this point is situated at altitude h_0 above sea level. The triple $\langle x, y, z\rangle$ will denote the coordinates of the aircraft center of gravity relative to C at some subsequent time t; ψ will denote the associated angle between the projection of the velocity vector on the x–y plane and the x axis. The subscript 0 will be universally used to imply the value of the corresponding parameter at the beginning of the turn; thus, $x_0 = y_0 = z_0 = \psi_0 = t_0 = 0$ by definition.

Received 1 July 2004; revision received 15 October 2004; accepted for publication 15 October 2004.

*Senior Lecturer, Faculty of Aerospace Engineering.

In what follows we shall adopt a standard notation where g, p, ρ, and γ are the acceleration of gravity, air static pressure, air density, and the ratio of the specific heats; v and M are the true airspeed and the Mach number. Also,

$$h_e = h_0 - z + v^2/2g \tag{1}$$

is the energy altitude; and W, T, D, L, and $n = L/W$ are the weight, thrust, drag, lift, and load factor. As customary, lift C_L and drag C_D coefficients will be defined based on the wing area S, in particular,

$$C_L = 2nW/\rho v^2 S = 2nW/\gamma p M^2 S \tag{2}$$

Constant-Altitude Turns

Equations of Motion

Equations of motion for constant-altitude turns can be found in Ref. 1. In present notation they take on the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v \sin\psi \tag{3}$$

$$\frac{\mathrm{d}h_e}{\mathrm{d}t} = \frac{1}{2g}\frac{\mathrm{d}v^2}{\mathrm{d}t} = v\left(\frac{T}{W} - n\frac{C_D}{C_L}\right) \tag{4}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{g}{v}\sqrt{n^2-1} \tag{5}$$

Because the end of the turn is specified by turn angle ψ, rather then by time t, it is more convenient to use ψ as the independent variable. Moreover, because the time required to complete the turn is inconsequential for the course of the present analysis, changing the variables also reduces the number of equations to two:

$$\frac{\mathrm{d}y}{\mathrm{d}\psi} = \frac{v^2}{g}\frac{\sin\psi}{\sqrt{n^2-1}} \tag{6}$$

$$\frac{1}{v^2}\frac{\mathrm{d}v^2}{\mathrm{d}\psi} = \frac{2}{\sqrt{n^2-1}}\left(\frac{T}{W} - n\frac{C_D}{C_L}\right) \tag{7}$$

These immediately follow Eqs. (3) and (4) by Eq. (5). Equations (6) and (7) can be further simplified using definition (2) of the lift coefficient and its corollary

$$v^2\big/v_0^2 = M^2\big/M_0^2 = n/n_0 \tag{8}$$

applicable for constant-altitude–constant-lift-coefficient turns. The result is a pair of equations

$$\frac{\mathrm{d}y}{\mathrm{d}\psi} = \frac{2W}{\rho_0 g S C_L}\frac{n}{\sqrt{n^2-1}}\sin\psi \tag{9}$$

$$\frac{\mathrm{d}n}{\mathrm{d}\psi} = \frac{2n}{\sqrt{n^2-1}}\left(\frac{T}{W} - n\frac{C_D}{C_L}\right) \tag{10}$$

of which the second one is independent of the first and, hence, in principle, can be solved separately.

Indeed, subject to the initial condition at $\psi = 0$,

$$n = n_0 \tag{11}$$

Equation (10) can be solved analytically for constant thrust, weight, and aerodynamic coefficients; this solution may be found in the Appendix. In the context of energy-to-lateral-displacement exchange it is ineffective, however, because, being implicit in n, it does not allow analytical integration of Eq. (9) for the displacement. Moreover, constant thrust and drag coefficient assumptions are probably inapplicable at low supersonic Mach numbers (see Numerical Example section). An approximate solution allowing thrust and drag coefficient to vary during the turn is suggested next.

Approximate Solution

Recall that we seek a relation between the loss of energy and the lateral displacement. It has been tacitly implied that only a tiny bit of the energy altitude (few thousand feet out of few tens of thousands) can be used toward tightening of the turn and, therefore, finding the derivative of the lateral displacement with respect to the energy loss may provide an adequate answer to the question. However, if only the derivative is required, general solution of Eq. (10) is unnecessary. Rather, it can be confined to the case where the lift coefficient and, consequently, the load factor, airspeed, etc., differ only slightly during the turn from their maximal sustained values, hereafter denoted by the subscript s.

Thus, with δ the small parameter, we set

$$C_L = C_{L,s}(1+\delta) \tag{12}$$

and assume, subject to a posteriori verification, that for any $\psi \geq 0$

$$v(\psi) = v_0 + v_1(\psi)\delta + \cdots \tag{13}$$

$$n(\psi) = n_s + n_1(\psi)\delta + \cdots \tag{14}$$

$$y(\psi) = y_s(\psi) + y_1(\psi)\delta + \cdots \tag{15}$$

$$h_e(\psi) = h_1(\psi)\delta + \cdots \tag{16}$$

with the ellipses standing for higher-order terms with respect to δ. The derivative of the lateral displacement with respect to the energy loss will, eventually, follow the quantities marked by the subscript 1. Because the velocity and coordinates of the aircraft at the beginning of the turn are given,

$$v_1(0) = y_1(0) = h_1(0) = 0 \tag{17}$$

During a constant-altitude turn, the thrust depends only on airspeed; whereas the drag coefficient depends on airspeed and lift coefficient. Hence, Eqs. (12–16) also imply that for any $\psi \geq 0$

$$T(\psi) = T_0 + T_1(\psi)\delta + \cdots \tag{18}$$

$$C_D(\psi) = C_{D,s} + C_{D,1}(\psi)\delta + \cdots \tag{19}$$

where

$$T_1(\psi) = \left(\frac{\partial T}{\partial v}\right)_{v=v_0} v_1(\psi) \tag{20}$$

$$C_{D,1}(\psi) = \left(\frac{\partial C_D}{\partial v}\right)_{\substack{C_L=C_{L,s}\\ v=v_0}} v_1(\psi) + \left(\frac{\partial C_D}{\partial C_L}\right)_{\substack{C_L=C_{L,s}\\ v=v_0}} C_{L,s} \tag{21}$$

Here, the subscripted conditions are those at which the respective partial derivatives need to be evaluated. Consistent with the notation convention introduced at the beginning of this section, these conditions will be abbreviated hereafter by the subscript s.

Now, substitute Eqs. (12–16), (18) and (19) in Eqs. (2), (9), (10) and (1). Because the last four equations should hold for any δ (provided, of course, that the respective series in the first six equations converge) they need to be satisfied at each order of δ separately.[5] Thus, in particular,

$$n_s = \frac{\rho_0 v_0^2 S C_{L,s}}{2W} \tag{22}$$

$$\frac{\mathrm{d}y_s}{\mathrm{d}\psi} = \frac{v_0^2}{g}\frac{\sin\psi}{\sqrt{n_s^2-1}} \tag{23}$$

$$n_s = \frac{T_0}{W}\frac{C_{L,s}}{C_{D,s}} \tag{24}$$

at the zero-order, and

$$n_1 = n_s\left(1 + \frac{2v_1}{v_0}\right) \tag{25}$$

$$\frac{\mathrm{d}y_1}{\mathrm{d}\psi} = -\frac{v_0^2}{g}\frac{\sin\psi}{\sqrt{n_s^2-1}}\left(1 + \frac{1}{n_s^2-1}\frac{n_1}{n_s}\right) \tag{26}$$

$$\frac{\mathrm{d}n_1}{\mathrm{d}\psi} = 2\frac{n_s}{\sqrt{n_s^2-1}}\left(\frac{T_1}{W} - \frac{(n_1-n_s)C_{D,s} + n_s C_{D,1}}{C_{L,s}}\right) \tag{27}$$

$$h_1 = \frac{v_0^2}{2g}\left(\frac{n_1}{n_s} - 1\right) \tag{28}$$

at the first order. Equations (23) and (26) have been obtained with the help of Eq. (22); Eq. (27) has been obtained with the help of Eq. (24); Eq. (28) has been obtained with the help of Eq. (25). After substitution of Eqs. (20) and (21) for T_1 and $C_{D,1}$, Eq. (27) needs additional attention. When Eqs. (25) and (24) are used to eliminate v_1 and T_0, it can be eventually recast into the form

$$\frac{\mathrm{d}n_1}{\mathrm{d}\psi} = \lambda_s(a - n_1) \tag{29}$$

where

$$a = n_s\left(1 - \frac{\lambda_1}{\lambda_s}\right) \tag{30}$$

$$\lambda_s = 2\frac{C'_{D,s}}{C_{L,s}}\frac{n_s}{\sqrt{n_s^2-1}} \tag{31}$$

$$\lambda_1 = 2\frac{n_s}{\sqrt{n_s^2-1}}\left(\frac{\partial C_D}{\partial C_L}\right)_s \tag{32}$$

$$C'_{D,s} = C_{D,s}\left[1 + \frac{v_0}{2C_{D,s}}\left(\frac{\partial C_D}{\partial v}\right)_s - \frac{v_0}{2T_0}\left(\frac{\partial T}{\partial v}\right)_s\right] \tag{33}$$

are convenient dimensionless combinations of model parameters.

Thus, it is concluded that n_s and $C_{L,s}$ are solutions of Eqs. (22) and (24), y_s and n_1 follow straightforward integrations in Eqs. (23) and (29), and y_1 follows an integration in Eq. (26) using the result for n_1. Explicitly, for any $\psi \geq 0$,

$$y_s(\psi) = \frac{v_0^2}{g}\frac{1-\cos\psi}{\sqrt{n_s^2-1}} \tag{34}$$

$$y_1(\psi) = -\frac{v_0^2}{g}\frac{1}{\sqrt{n_s^2-1}}\left\{\left(1 + \frac{1}{n_s^2-1}\frac{a}{n_s}\right)(1-\cos\psi) + \left(\frac{1}{n_s^2-1}\frac{n_s-a}{n_s}\frac{1}{1+\lambda_s^2}\right)\left[1 - e^{-\lambda_s\psi}(\cos\psi + \lambda_s\sin\psi)\right]\right\} \tag{35}$$

$$n_1(\psi) = a + (n_s - a)e^{-\lambda_s\psi} \tag{36}$$

To obtain Eq. (36), we have used initial condition $n_1(0) = n_s$, stemming from Eqs. (25) and (17). The expressions for $v_1(\psi)$ and $h_1(\psi)$ follow Eqs. (25) and (28) by Eq. (36).

Conditions at the exit of the turn are those at $\psi = \pi$, in particular,

$$y_{s,\pi} = y_s(\pi) = \frac{2v_0^2}{g}\frac{1}{\sqrt{n_s^2-1}} \tag{37}$$

$$y_{1,\pi} = y_1(\pi) = -\frac{2v_0^2}{g}\frac{n_s^2}{\left(n_s^2-1\right)^{\frac{3}{2}}}\left(1 - \frac{1}{n_s^2}\frac{\lambda_1}{\lambda_s}\frac{1+2\lambda_s^2-e^{-\lambda_s\pi}}{2+2\lambda_s^2}\right) \tag{38}$$

$$h_{1,\pi} = h_1(\pi) = -\frac{v_0^2}{2g}\frac{\lambda_1}{\lambda_s}(1 - e^{-\lambda_s\pi}) \tag{39}$$

Given that $\lambda_s > 0$, Eq. (36) implies that a new sustained load factor $n_s + a\delta$, corresponding to new lift coefficient $C_{L,s} + C_{L,s}\delta$, is set exponentially with turn angle ψ. The power of this exponent depends on the load factor and on the effective drag to lift coefficients ratio $C'_{D,s}/C_{L,s}$. Both dependences are plausible: On the one hand, the higher the drag is, the faster should the extra energy dissipate and stabilized conditions be set; on the other hand, the lower the load factor is, the slower the turn rate is, and hence, the smaller is the turn angle on which the turn rate eventually stabilizes. Thrust increase with airspeed or drag coefficient decrease with airspeed (at constant lift coefficient) decreases the effective drag coefficient $C'_{D,s}$ and, hence, increases the turn setting time.

The ratio of Eqs. (38) and (39) yields, of course, the derivative of the lateral displacement with respect to the energy loss

$$\left(\frac{\mathrm{d}y_\pi}{\mathrm{d}h_{e,\pi}}\right)_s = \lim_{\delta\to 0}\frac{y(\pi) - y_s(\pi)}{h_e(\pi) - 0} = \frac{y_{1,\pi}}{h_{1,\pi}} = \frac{4}{\pi\lambda_1}\frac{n_s^2}{\left(n_s^2-1\right)^{\frac{3}{2}}}\frac{\pi\lambda_s}{1-e^{-\lambda_s\pi}}\left(1 - \frac{1}{n_s^2}\frac{\lambda_1}{\lambda_s}\frac{1+2\lambda_s^2-e^{-\lambda_s\pi}}{2+2\lambda_s^2}\right) \tag{40}$$

The discussion follows later.

Constant Mach Turns

Equations of Motion

Consider now the case where Mach number, rather then altitude, is kept constant during the turn. To assist the solution, it will be assumed that the turn is executed in the stratosphere, where temperature is independent of altitude; hence, constant Mach number implies constant airspeed throughout the turn. The solution itself will be analogous to that for the constant-altitude turns, but the assumption that the lift coefficient during the turn differs only slightly from its sustained value will be introduced as early as during the statement of the equations of motion.

We begin with the energy equation,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -v_0\left(\frac{T}{W} - n\frac{C_D}{C_L}\right) \tag{41}$$

which, considering Eq. (1), is tantamount to Eq. (4). Besides the obvious objective of manipulating this equation into a solvable form, the idea is to use it for setting the order with respect to δ of the trajectory-angle-dependent terms in the remaining equations,[1] which will be addressed later on.

It immediately follows from Eq. (2) that

$$n/n_0 = p/p_0 \tag{42}$$

Indeed, for constant-Mach–constant-lift-coefficient turns all quantities in Eq. (2) besides p and n are constant by assumption and equal their respective values at the beginning of the turn. At the same time, in the stratosphere,[6]

$$p/p_0 = \rho/\rho_0 = \exp(\rho_0 g z/p_0) \tag{43}$$

accordingly on the left-hand side of Eq. (41),

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{p_0}{\rho_0 g}\frac{1}{n}\frac{\mathrm{d}n}{\mathrm{d}t} \tag{44}$$

On the right-hand side of Eq. (41), thrust variation with altitude is customary modeled by[6]

$$T = T_0(p/p_0) \tag{45}$$

whereas drag coefficient variation with altitude is customarily ignored altogether. [Weak dependence of the drag coefficient on altitude is related to the dependence of skin-friction coefficient on Reynolds number (see Ref. 4).]

With Eqs. (42–45), Eq. (41) takes on the form

$$\frac{1}{n^2}\frac{\mathrm{d}n}{\mathrm{d}t} = -\frac{\rho_0 g v_0}{p_0 n_0}\left(\frac{T_0}{W} - n_0\frac{C_D}{C_L}\right) \tag{46}$$

where the right-hand side is independent of time. In conjunction with Eq. (44), it can be used to compute the vertical acceleration,

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = -\frac{\rho_0 g}{p_0}\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 \tag{47}$$

Alternatively, it can be readily integrated to obtain n and, hence, z as functions of time. The result of this integration is quite obvious, but useless in the present context.

At this point, we would like to introduce the assumption that the lift coefficient during the turn differs only slightly from its sustained value. Thus, when δ is assumed small compared with unity, we substitute Eqs. (12), (14), and (19) in Eq. (46). It becomes immediately apparent that its right-hand side is of order δ; hence $\mathrm{d}z/\mathrm{d}t$ is of order δ by Eq. (44), and hence, $\mathrm{d}^2z/\mathrm{d}t^2$ is of order δ^2 by Eq. (47). These imply that in the leading order with respect to δ, the remaining equations of motion (pp. 101–102, Ref. 1) are identical with Eqs. (3) and (5) describing constant-altitude turns. Hence, confining the discussion only to the leading order with respect to δ, Eq. (5) can be introduced in Eq. (3) to recover Eq. (9); in turn, it can be introduced in Eq. (46) to obtain

$$\frac{\mathrm{d}n}{\mathrm{d}\psi} = -\frac{n^2}{\sqrt{n^2-1}}\frac{\rho_0 v_0^2}{p_0 n_0}\left(\frac{T_0}{W} - n_0\frac{C_D}{C_L}\right) + \cdots \tag{48}$$

Approximate Solution

Asymptotic solution of Eqs. (9) and (48) is identical, mutatis mutandis, with the solution of Eqs. (9) and (10) for the constant-altitude case: First, introduce Eqs. (12), (14–16), and (19) in Eqs. (9) and (48), and then require the resulting equations to be satisfied at each order with respect to δ. This procedure recovers Eqs. (23) and (24) in the zeroth order and yields

$$\frac{\mathrm{d}y_1}{\mathrm{d}\psi} = -\frac{v_0^2}{g}\frac{n_1 n_s}{\left(n_s^2-1\right)^{\frac{3}{2}}}\sin\psi \tag{49}$$

$$\frac{\mathrm{d}n_1}{\mathrm{d}\psi} = \gamma M_0^2 \frac{n_s^2}{\sqrt{n_s^2-1}}\left(\frac{\partial C_D}{\partial C_L}\right)_s = \frac{1}{2}\gamma M_0^2 n_s \lambda_1 \tag{50}$$

in the first order.

For constant lift coefficient and constant Mach number, the right-hand side of Eq. (50) remains constant during the turn. Moreover, right at the beginning of the turn, $n_1(0) = n_s$ by Eqs. (2), (12), and (14). Hence, for any $\psi \geq 0$,

$$n_1(\psi) = n_s + \tfrac{1}{2}\gamma M_0^2 n_s \lambda_1 \psi \tag{51}$$

With Eq. (51), the integration in Eq. (49) is immediate; subject to Eq. (17) it yields

$$y_1(\psi) = -\left(v_0^2/g\right)\left[n_s^2/\left(n_s^2-1\right)^{\frac{3}{2}}\right]$$
$$\times\left[1 - \cos\psi + \tfrac{1}{2}\gamma M_0^2\lambda_1(\sin\psi - \psi\cos\psi)\right] \tag{52}$$

The associated change in the energy altitude

$$h_1(\psi) = -(p_0/\rho_0 g)[n_1(\psi) - n_s]/n_s \tag{53}$$

follows Eqs. (1) and (44) by Eq. (16). Thus, at the exit of the turn,

$$y_{1,\pi} = y_1(\pi) = -\frac{2v_0^2}{g}\frac{n_s^2}{\left(n_s^2-1\right)^{\frac{3}{2}}}\left(1 + \frac{\pi}{4}\gamma M_0^2\lambda_1\right) \tag{54}$$

$$h_{1,\pi} = h_1(\pi) = -\frac{\pi}{2}\frac{\gamma M_0^2 p_0}{\rho_0 g}\lambda_1 = -\frac{\pi}{2}\frac{v_0^2}{g}\lambda_1 \tag{55}$$

and, consequently,

$$\left(\frac{\mathrm{d}y_\pi}{\mathrm{d}h_{e,\pi}}\right)_{C_L = C_{L,s}} = \frac{y_{1,\pi}}{h_{1,\pi}} = \frac{4}{\pi\lambda_1}\frac{n_s^2}{\left(n_s^2-1\right)^{\frac{3}{2}}}\left(1 + \frac{\pi}{4}\gamma M_0^2\lambda_1\right) \tag{56}$$

Numerical Example

Model Aircraft

To elucidate the theory developed in the preceding two sections, consider a hypothetical aircraft roughly resembling the F15C in shape, dimensions [wing area of 56.5 m^2 (609 ft^2)], thrust, and weight. Thrust and drag models of this hypothetical aircraft with three 600-gallon (2300 liter) external tanks have been constructed using preliminary design tools of Ref. 4. Neither model has the pretension of accurately representing the real F15C; nonetheless, each is complex enough to challenge the assumptions of this exposition.

The thrust variation with the Mach number and altitude is shown in the left-hand side of Fig. 1. The drag coefficient was assumed to possess the form

$$C_D = C_{D,p} + KC_L^2 \tag{57}$$

where parasite (zero lift) drag coefficient $C_{D,p}$ is a function of the Mach number only, whereas, induced drag parameter K is a function of both the Mach number and the lift coefficient. $C_{D,p}$ and K are shown on the right-hand sides in Fig. 1. Also, shown in Fig. 1 is the lift-slope coefficient $C_{L,\alpha}$, whose estimation was a prerequisite for estimation of K.

Setup and Validation

Consider now a full-throttle sustained level turn at 12,192-m (40,000-ft) altitude in standard atmosphere, once at Mach 0.8 and once at Mach 1.2. With aircraft weighing 239.6 kN (53,850 lb), these turn conditions will be referred to as A and B. The model aircraft is capable of sustaining 1.45-g turn with diameter of about 10,900 m (5.9 n mile) at A and 1.6-g turn with diameter of about 20,500 m (11 n mile) at B. Thrust, lift, and drag coefficients at these sustained conditions are marked in Fig. 1 by small circles; additional data, including pertinent thrust and drag coefficient derivatives, are listed in Table 1.

Increasing the lift coefficient above its sustained value tightens the turn and takes its energy toll; conversely, reducing the lift coefficient below its sustained value widens the turn and restores energy. For small variations in the lift coefficient, the respective effects can be estimated with Eqs. (13–16), (25), (28), (35), (36), (38), and (39) for constant-altitude turns and with Eqs. (14–16), and (51–54) for constant Mach turns. Based on data of Table 1, pertinent results are reported in Tables 2 and 3 and Figs. 2 and 3.

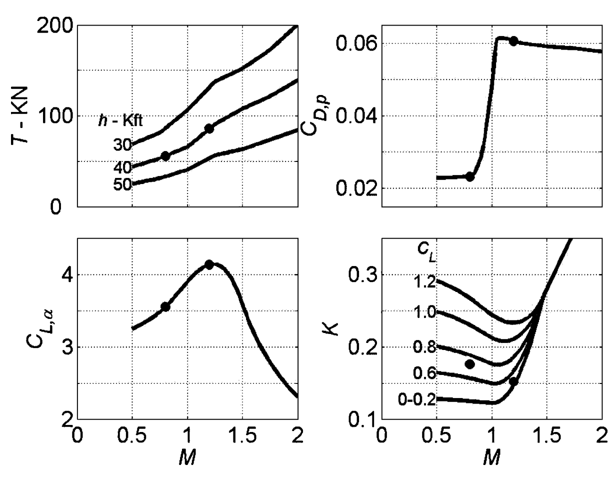

Fig. 1 Estimated full-throttle thrust, parasite-drag coefficient, induced-drag parameter, and lift-slope coefficient of 250-kN (55,000-lb) class supersonic aircraft: ●, example conditions.

Table 1 Pertinent model parameters at 239.6-kN (53,850-lb) gross weight, 12,192-m (40,000-ft) altitude, Mach 0.8 (condition A) and Mach 1.2 (condition B)

Parameter	Condition A	Condition B	Parameter	Condition A	Condition B
n_s	1.446	1.600	K	0.176	0.158
$C_{L,\alpha}$	3.55	4.13	$(\partial C_D/\partial C_L)_s$	0.345	0.114
$C_{L,s}$	0.730	0.359	$(v_0/C_{D,s})(\partial C_D/\partial v)_s$	−0.129	0.325
$C_{D,s}$	0.117	0.080	T_0/W	0.232	0.357
$C_{D,p}$	0.0233	0.0606	$(v_0/T_0)(\partial T/\partial v)_s$	0.719	1.436

Table 2 Constant altitude turns

	$y_{1,\pi}$, m		$h_{1,\pi}$, m		$(\mathrm{d}y_\pi/\mathrm{d}h_{e,\pi})_s$	
Condition	Eq. (38)	Simulation	Eq. (39)	Simulation	Eq. (40)	Simulation
$M_0 = 0.8$	−8905	−8881	−5862	−5880	1.52	1.51
$M_0 = 1.2$	−28760	−28750	−4050	−4051	7.10	7.10

Table 3 Constant Mach turns

	$y_{1,\pi}$, m		$h_{1,\pi}$, m		$(\mathrm{d}y_\pi/\mathrm{d}h_{e,\pi})_s$	
Condition	Eq. (54)	Simulation	Eq. (55)	Simulation	Eq. (56)	Simulation
$M_0 = 0.8$	−34894	−34844	−8528	−8521	4.09	4.09
$M_0 = 1.2$	−49148	−49100	−5881	−5877	8.36	8.36

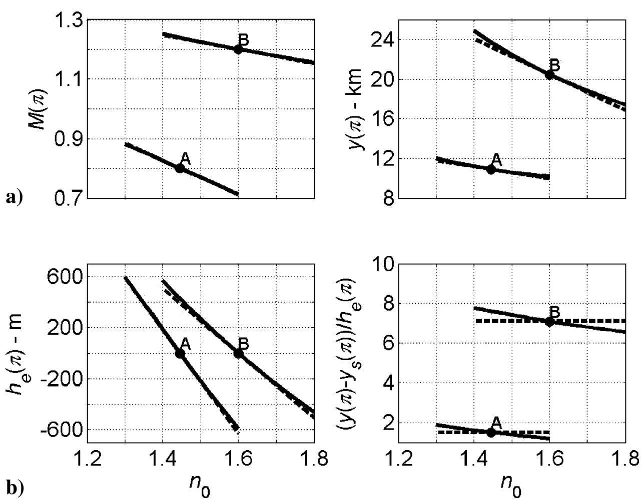

Fig. 2 Constant altitude turns at conditions A and B at various load factors: ——, numerical solution of Eqs. (3–5); – – –, estimates based on Eqs. (25), (28), (35), and (36); and ●, sustained turn conditions.

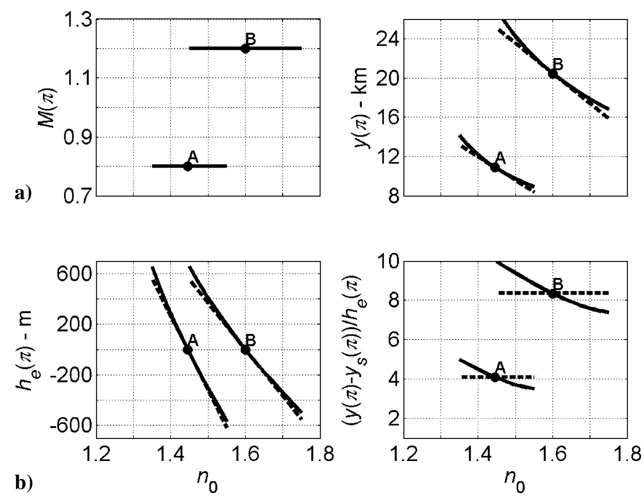

Fig. 3 Constant Mach turns at conditions A and B at various load factors: ——, direct numerical solution of Eqs. (3), (5), and (41); – – –, estimates based on Eqs. (51–53); and ●, sustained turn conditions.

It is implicitly understood that lateral displacement and energy altitude at the exit of a turn can be obtained by solving the respective equations of motion numerically. These are Eqs. (3–5) for constant altitude turns and Eqs. (3), (5), and (41) for small-altitude-loss constant Mach turns. We did solve them for initial conditions based on conditions A and B and lift coefficient ranging from about 10% below to 10% above the corresponding sustained value; pertinent results are reported in Tables 2 and 3 and Figs. 2 and 3 along with their small-perturbations counterparts. The agreement between the two is adequate for the entire range of lift coefficients tested.

Results

With reference to Tables 2 and 3, notice that, at both flight conditions tested, constant-altitude turns are wider (have smaller values of $y_{1,\pi}$) and require less energy (have smaller values of $h_{1,\pi}$) to complete. This result will be discussed in the next section. At the same time, when the right columns in Tables 2 and 3 are examined, constant-Mach turns are superior (have larger values of the derivative $\mathrm{d}y_\pi/\mathrm{d}h_\pi$) to constant-altitude turns insofar as the energy-to-lateral-displacement exchange is concerned. In particular, a constant-Mach turn at condition B tightens about 840 m for every 100 m of energy altitude lost, whereas the comparable constant-altitude turn tightens only 710 m.

At this altitude and Mach, every 100-m loss of energy altitude are equivalent to 0.0094 loss in Mach number. Accordingly, 0.03 Mach lost in a constant-altitude turn, or 270 m (900 ft) lost in a constant Mach turn, should have reduced the lateral displacement in the turn by about 2300 m (1.2 n mile): from the sustained 11 n mile down to less than 10 n mile. In the context of the flight-test problem that triggered this study (see Background section), this is a huge effect.

With reference to Figs. 2 and 3, its is quite apparent that the small-perturbations solution is adequate for all practical purposes in predicting Mach, altitude, and lateral displacement at the exit of a turn. The prediction error grows with the perturbation, an intrinsic property of any leading-order solution. The prediction error is the most apparent in the right-hand sides of Figs. 2 and 3, where the ratio $y_{1,\pi}/h_{1,\pi}$, constant by definition, is displayed against the ratio of the simulated (exact) lateral displacement and energy altitude variations.

Discussion

Stationary Variations

The combinations

$$Y = -\left(2v_0^2/g\right)\left[n_s^2/\left(n_s^2-1\right)\right]^{\frac{3}{2}} \tag{58}$$

$$H = -(\pi/2)\left(v_0^2/g\right)\lambda_1 \tag{59}$$

manifested on the right-hand sides of Eqs. (38), (54), (39), and (55), can be identified with the variations in lateral displacement and in specific energy required to complete the turn under the assumption that the Mach number, altitude, and load factor are kept constant throughout the turn. This conjecture can be verified by direct integration in Eqs. (3–5), followed by differentiation with respect to C_L. In other words, Y and H are the respective variations of under the constraint that thrust equals drag. The former is a universal quantity depending on flight conditions only; the latter can be extracted from specific excess power figures found in most operators' manuals.

Nonstationary Effects

With no extra thrust available to compensate for the increased drag, either the kinetic or the potential energy suffers, and hence, the specific energy required to complete the turn becomes the energy altitude lost during the maneuver. However, because the load factor and the flight conditions now change during the turn, neither Y nor H represents correctly the respective variation. The nonstationary effects of energy depleting turns are manifested in the terms that multiply Y in Eqs. (38) and (54) and H in Eqs. (39) and (55); in what follows they will be marked by an overbar. Specifically,

$$\bar{y}_{1,\pi} = \frac{y_{1,\pi}}{Y} = 1 - \frac{1}{n_s^2}\frac{\lambda_1}{\lambda_s}\frac{1+2\lambda_s^2-e^{-\lambda_s\pi}}{2+2\lambda_s^2} \tag{60}$$

$$\bar{h}_{1,\pi} = \frac{h_{1,\pi}}{H} = \frac{1-e^{-\lambda_s\pi}}{\pi\lambda_s} \tag{61}$$

for constant-altitude turns and

$$\bar{y}_{1,\pi} = 1 + (\pi/4)\gamma M_0^2\lambda_1 \tag{62}$$

$$\bar{h}_{1,\pi} = 1 \tag{63}$$

for constant-Mach turns.

The magnitudes of the nonstationary effects depend on particular combinations of n_s, M_0, λ_1, and λ_s. Because the last two parameters are model dependent, no general conclusions can be made. For the model aircraft of the preceding section at condition B, $Y \approx 33{,}600$ m (18.1 n mile) by Eq. (58), whereas $H \approx 5900$ m (19,300 ft) by Eq. (59). Hence, $\bar{y}_1 \approx 0.85$ and $\bar{h}_1 \approx 0.69$ for constant-altitude turns and $\bar{y}_1 \approx 1.44$ and $\bar{h}_1 = 1$ for constant-Mach turns. (To obtain these numbers, divide the first two columns in the last line of each of Tables 2 and 3 by Y and the next two columns by H.) These effects are obviously too large to be ignored.

Lateral Displacement

The expression $(1/\lambda_s)[(1+2\lambda_s^2-e^{-\lambda_s\pi})/(2+2\lambda_s^2)]$, appearing in Eq. (60), is positive for any λ_s, positive and negative alike. As long as no drag anomalies are involved, λ_1 is positive as well, compare Eq. (32). Hence, $\bar{y}_{1,\pi}$ is less than unity for constant-altitude turns and greater than unity for constant-Mach turns, compare Eq. (62). An immediate implication of these relations is that energy-depleting constant-altitude turns are always wider than their constant-Mach counterparts.

Because the load factor is the sole parameter (lift coefficient kept constant) governing lateral displacement during both constant-altitude and small-trajectory-angle constant-Mach turns [see Eq. (9)], its behavior explains this intrinsic difference between the two turn strategies. In particular, the load factor diminishes as the airspeed diminishes during constant-altitude turns and increases as the altitude decreases during constant-Mach turns. Diminishing load factor widens the turn; increasing load factor tightens it.

When the effect of diminishing airspeed in a constant-altitude turn is taken to the extreme, it is conceivable that for some conditions the loss of airspeed during the turn can become large enough to make $\bar{y}_{1,\pi}$ negative. For those conditions, $y_{1,\pi}$ becomes positive, and hence, an increase in the lift coefficient acts to increase the lateral displacement, rather than to decrease it.

This conjecture can be supported considering a constant-altitude turn executed just below the service ceiling,[6] that is, where the maximum sustained load factor is barely above unity. Taking the limit $n_s \to 1$ in Eq. (60), one finds

$$\lim_{n_s\to 1}\bar{y}_{1,\pi} = 1 - \frac{C_{L,s}}{C'_{D,s}}\left(\frac{\partial C_D}{\partial C_L}\right)_s \tag{64}$$

by Eqs. (31) and (32). Apparently, $\bar{y}_{1,\pi}$ changes sign (turns negative) when the rightmost term becomes greater than unity.

If the drag coefficient can be approximated by a parabolic polar with constant coefficients, and the thrust can be approximated as independent of the Mach number, then $\bar{y}_{1,\pi} < 0$ when $C_{L,s} > \sqrt{(C_{D,p}/K)}$, or, using pilots' jargon, at the backside of the thrust required curve.

Energy Loss

From Eq. (63), $h_{1,\pi}$ is unity for constant-Mach turns. From Eq. (61), its constant-altitude turns counterpart is less than unity only for positive values of λ_s. Under most circumstances, λ_s is indeed positive; hence, constant-altitude turns typically lose less energy than comparable constant-Mach turns. Still, if thrust increases rapidly enough with the Mach number, λ_s may turn negative, compare Eq. (31). In this case, an increase in the lift coefficient yields an exponentially fast decay of airspeed, compare Eqs. (25) and (36), and hence the excessive energy loss; the turn "chocks" itself.

Conclusions

With use of a perturbation technique, approximate analytical expressions have been obtained for the load factor, airspeed, altitude, lateral displacement, and energy loss during constant-altitude and constant-Mach turns. The accuracy of these approximations was verified by direct numerical simulations and found adequate. It is believed that the same technique can be used to address large variety of conceptually similar problems.

It was shown that constant-altitude turns are always wider than comparable constant Mach turns; under most circumstances, they also lose less energy. Nonetheless, no a priori conclusion can be made which of the two turn strategies is more effective in exchanging energy for lateral displacement. For the model aircraft in particular flight conditions used in this study, constant-Mach turns were more effective.

Appendix: Exact Solution of Equation (10)

Let a, b, and n_0 be three real constants. It is assumed that $a \geq 0$ and $n_0 > 1$; no restrictions are imposed on b. We seek the solution of the differential equation

$$\frac{\mathrm{d}n}{\mathrm{d}\psi} = \frac{2bn}{\sqrt{n^2-1}}(a-n) \tag{A1}$$

subject to the initial condition at $\psi = 0$

$$n = n_0 \tag{A2}$$

In view of the particular form of Eq. (A1), this solution can always be written in the form

$$2b\psi = L(n,a) - L(n_0,a) \tag{A3}$$

where L is a function on $(1, \infty) \times [0, \infty)$ such that for any n in $(1, \infty)$ and a in $[0, \infty)$,

$$L(n, a) = \int_1^n \frac{\sqrt{m^2 - 1}}{m(a - m)}\, \mathrm{d}m \tag{A4}$$

L can be evaluated analytically using standard substitution

$$t = \sqrt{(m - 1)/(m + 1)}$$

The result is

$$L(n, a) = \ln\left(\frac{\sqrt{n+1} - \sqrt{n-1}}{\sqrt{n+1} + \sqrt{n-1}}\right) - \frac{1}{a}\cos^{-1}\frac{1}{n} + 2\frac{\sqrt{1 - a^2}}{a}\tan^{-1}\left(\sqrt{\frac{n-1}{n+1}}\sqrt{\frac{1+a}{1-a}}\right) \tag{A5}$$

for each a in $[0, 1]$ and

$$L(n, a) = \ln\left(\frac{\sqrt{n+1} - \sqrt{n-1}}{\sqrt{n+1} + \sqrt{n-1}}\right) - \frac{1}{a}\cos^{-1}\frac{1}{n} - \frac{\sqrt{a^2 - 1}}{a}\ln\left|\frac{\sqrt{n+1}\sqrt{a-1} - \sqrt{n-1}\sqrt{a+1}}{\sqrt{n+1}\sqrt{a-1} + \sqrt{n-1}\sqrt{a+1}}\right| \tag{A6}$$

for each a in $[1, \infty)$. Note that for each a in $[0, \infty)$

$$L(1, \alpha) = 0 \tag{A7}$$

by definition, compare (A4).

Useful particular cases are when $|n - a|$ is small and, of course, both n and a greater than unity and when a either equals unity or is small compared with unity. Formal expansions of Eqs. (A5) and (A6) corresponding to these cases, in that order, are

$$L(n, a) = L_0(a) - \frac{\sqrt{a^2 - 1}}{a}\ln|n - a| + \mathcal{O}(|n - a|) \tag{A8}$$

$$L(n, 1) = \ln\left(\frac{\sqrt{n+1} - \sqrt{n-1}}{\sqrt{n+1} + \sqrt{n-1}}\right) - \cos^{-1}\frac{1}{n} \tag{A9}$$

$$L(n, a) = \ln\left(\frac{\sqrt{n+1} - \sqrt{n-1}}{\sqrt{n+1} + \sqrt{n-1}}\right) + \frac{\sqrt{n^2 - 1}}{n} + \frac{a}{2}\left(\frac{\sqrt{n^2 - 1}}{n^2} - \cos^{-1}\frac{1}{n}\right) + \mathcal{O}(a^2) \tag{A10}$$

Here,

$$L_0(a) = \ln\left(\frac{\sqrt{a+1} - \sqrt{a-1}}{\sqrt{a+1} + \sqrt{a-1}}\right) - \frac{1}{a}\cos^{-1}\frac{1}{a} + \frac{\sqrt{a^2 - 1}}{a}\ln(2a^2 - 2) \tag{A11}$$

is the regular part of $L(a, a)$; it is immaterial for the solution (A3) because it cancels out in due course. Formal expansion at $a = 1$ is singular as n goes to unity and, hence, is useless.

References

[1]Vinh, N. X., *Optimal Trajectories in Atmospheric Flight*, Elsevier Scientific, Amsterdam, 1981, pp. 101–130.

[2]Gates, S. B., "A Study of Aircraft Turning Performance. Part 1," Royal Aircraft Establishment, RAE Reports and Memoranda, No. 1502, London, June 1932.

[3]Etkin, B., *Dynamics of Atmospheric Flight. Stability and Control*, Wiley, New York, 1972, pp. 239, 513, 532.

[4]Raymer, D., *Aircraft Design. A Conceptual Approach*, AIAA Educational Series, AIAA, Washington, DC, 1995, pp. 264, 265, 278–296, 298–302, 315–317, 718.

[5]Van Dyke, M., *Perturbation Methods in Fluid Mechanics*, Parabolic, Stanford, CA, 1975, pp. 7–14.

[6]Vinh, N. X., *Flight Mechanics of High-Performance Aircraft*, Cambridge Univ. Press, Cambridge, England, U.K. 1993, pp. 29, 76, 191.